7 Differential Equations

7.1 Definitions and terminologies

An **(ordinary) differential equation** (or **ODE**) is an equation containing a function of one independent variable and its derivatives.

In general, an ODE of **order** n takes the form:

$$F\left(x, y, y', y'', \cdots, y^{(n-1)}\right) = y^{(n)}, \qquad (*)$$

where F is a given multi-variate function of x, y and derivatives of y, and $y^{(n)}$ denotes the nth derivative of y with respect to x.

A differential equation independent of x is called **autonomous**.

A differential equation is said to be **linear** if the function F in $(*)$ can be written as a linear combination of the derivatives of y:

$$y^{(n)} = \sum_{i=0}^{n-1} a_i(x) y^{(i)} + r(x),$$

where $a_i(x)$ and $r(x)$ are continuous functions in x. When the function $r(x)$ vanishes, the differential equation is called **homogeneous**.

Given an ODE in the form of $(*)$, a function u is called a **solution** of $(*)$, if u and its derivatives satisfy

$$F\left(x, u, u', u'', \cdots, u^{(n-1)}\right) = u^{(n)}.$$

A **general solution** of an nth-order equation is a solution containing n arbitrary independent constants of integration. A **particular solution** is derived from the general solution by setting the constants to particular values, often chosen to fulfill set "**initial conditions**" or "**boundary conditions**".

Here are some examples of ODEs and their solutions:

1. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy$; first-order, linear, homogeneous.
   The general solution is: $y = C\mathrm{e}^{\frac{1}{2}x^2}$.
   Given that the curve passes through the point $(1, \mathrm{e})$, the particular solution is: $y = \mathrm{e}^{\frac{1}{2}(x^2+1)}$.

2. $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^2 = a^2$, where $a > 0$ is a constant; first-order, non-linear, autonomous.
   The general solution is: $y = a\sin(x - x_0)$.

3. $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$; third-order, linear, autonomous, homogeneous.
   The general solution is: $y = C_1\cos 2x + C_2\sin 2x + C_3$.

4. $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + (x^2 + 4x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + (2x + 2)y = \mathrm{e}^{-x}$; second-order, linear.
   The general solution is: $y = \dfrac{1}{x^2}\left(C_1 + C_2\mathrm{e}^{-x} - x\mathrm{e}^{-x}\right)$.

5. $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = \ln x + 1$; second-order, linear.
   The general solution is: $y = C_1x + \dfrac{C_2}{x} - \ln x - 1$.
   Given the initial conditions: when $x = 1$, $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$, the particular solution is: $y = 2x - \dfrac{1}{x} - \ln x - 1$.

Exercise 45.

Verify that the above solutions satisfy the respective ODEs.

7.2 Separable first-order ODEs

ODE	Method	General Solution
$\dfrac{dy}{dx} = F(x)$	Direct integration with respect to x	$y = \int F(x)\,dx$
$\dfrac{dy}{dx} = G(y)$	Direct integration with respect to y	$x = \int \dfrac{1}{G(y)}\,dy$
$\dfrac{dy}{dx} = F(x)G(y)$	Separation of variables	$\int \dfrac{1}{G(y)}\,dy = \int F(x)\,dx$

Exercise 46.

1. Find the general solution of each of the following differential equations:
   (a) $\dfrac{dy}{dx} = \dfrac{x^2}{x-1}$
   (b) $\dfrac{dy}{dx} = \dfrac{y^2}{\ln y}$
   (c) $\dfrac{dy}{dx} = \dfrac{x^2}{y}$
   (d) $\dfrac{du}{dt} = u\tan t$
   (e) $\dfrac{dy}{dx} = \dfrac{xe^x(y^2+1)}{2y}$
   (f) $e^y\dfrac{dy}{dx} - 1 = \ln x$
   (g) $w\dfrac{dw}{dt} = (1-w)^2 t\sin t$
2. Find the solution of the differential equation $\dfrac{dy}{dx} = xye^{2x}$ for which $y = 1$ when $x = 0$.
3. Find the solution of the differential equation $x\dfrac{dy}{dx} = 2y - 1$ for which $y = 1$ when $x = 1$.
4. A curve is such that its gradient at any point P is the square root of the gradient of the line OP, where O is the origin. It is given that the curve passes through the point $(1, 4)$, express y in term of x.
5. In a given region, the population of a certain species is denoted by x. The birth rate (number of births per unit time) of the species is proportional to $\sqrt{x}$, and the death rate (number of deaths per unit time) of the species is proportional to x. At the beginning, $x = 100$, the birth rate is 3, and the death rate is 2.
   (a) Establish a differential equation involving x and t (time) to model the growth of the species.
   (b) Obtain an expression for x in terms of t.
   (c) Explain what happens to the population of the species in a long run.
6. (a) By considering $u = ye^x$, solve the differential equation

$$\frac{dy}{dx} + y = x,$$

   for which $y = 0$ when $x = 1$.
   (b) (†) Suggest a method to find the general solution of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x),$$

   where $p(x)$ and $q(x)$ are functions of x.

Exercise 47.

(†) Enjoy the following challenges.

1. Fluid flows steadily under a constant pressure gradient along a straight tube of circular cross-section of radius a. The velocity v of a particle of the fluid is parallel to the axis of the tube and depends only on the distance r from the axis. The equation satisfied by v is
$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = -k,$$
where k is constant. Find the general solution for v.
Show that $|v| \to \infty$ as $r \to 0$ unless one of the constants in your solution is chosen to be 0.
Suppose that this constant is, in fact, 0 and that $v = 0$ when $r = a$. Find v in terms of k, a and r.
The volume F flowing through the tube per unit time is given by
$$F = 2\pi \int_0^a rv \, \mathrm{d}r.$$
Find F.

2. Show that $\sin(k \sin^{-1} x)$, where k is a constant, satisfies the differential equation
$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + k^2y = 0. \qquad (*)$$
In the particular case when $k = 3$, find the solution of equation $(*)$ of the form
$$y = Ax^3 + Bx^2 + Cx + D,$$
that satisfies $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ at $x = 0$. Use this result to express $\sin 3\theta$ in terms of powers of $\sin\theta$.

3. In a cosmological model, the radius R of the universe is a function of the age t of the universe. The function R satisfies the three conditions:
$$R(0) = 0, \qquad R'(t) > 0 \text{ for } t > 0, \qquad R''(t) < 0 \text{ for } t > 0. \qquad (*)$$
The function H is defined by
$$H(t) = \frac{R'(t)}{R(t)}.$$
   (a) Sketch a graph of $R(t)$. By considering a tangent to the graph, show that $t < \dfrac{1}{H(t)}$.
   (b) Observations reveal that $H(t) = \dfrac{a}{t}$, where a is constant. Derive an expression for $R(t)$.
   What range of values of a is consistent with the three conditions $(*)$?
   (c) Suppose, instead, that observations reveal that $H(t) = \dfrac{b}{t^2}$, where b is constant. Show that this is not consistent with conditions $(*)$ for any values of b.

4. Given that $y = x$ and $y = 1 - x^2$ satisfy the differential equation
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0, \qquad (*)$$
show that $p(x) = \dfrac{2x}{1 + x^2}$ and $q(x) = \dfrac{2}{1 + x^2}$.
Show also that $ax + b(1 - x^2)$ satisfies the differential equation for any constants a and b.
Given instead that $y = \cos^2 \dfrac{x^2}{2}$ and $y = \sin^2 \dfrac{x^2}{2}$ satisfy the equation $(*)$, find $p(x)$ and $q(x)$.

5. A liquid of fixed volume V is made up of two chemicals A and B. A reaction takes place in which A converts to B. The volume of A at time t is xV and the volume of B at time t is yV where x and y depend on t and $x + y = 1$. The rate at which A converts into B is given by $kVxy$, where k is a positive constant. Show that if both x and y are strictly positive at the start, then at time t
$$y = \frac{D\mathrm{e}^{kt}}{1 + D\mathrm{e}^{kt}},$$
where D is a constant. Does A ever completely convert to B? Justify your answer.

6. Show that, if $y^2 = x^k f(x)$, then $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky^2 + x^{k+1}f'(x)$.

(a) By setting $k = 1$ in this result, find the solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + x^2 - 1$$

for which $y = 2$ when $x = 1$. Describe geometrically this solution.

(b) Find the solution of the differential equation

$$2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln x - xy^2$$

for which $y = 1$ when $x = 1$.

7. (a) The gradient y' of a curve at a point (x, y) satisfies

$$(y')^2 - xy' + y = 0. \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.
Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

(b) The gradient y' of a curve at a point (x, y) satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0.$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

8. Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0. \qquad (**)$$

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

9. (a) Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad (x \geq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0.$$

Hence show that the general solution can be written in the form $x^2(y^2 - x^2) = C$, where C is a constant.

(b) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 6y = 0 \qquad (x \neq 0).$$

10. (a) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is $y = x\sqrt{4 + 2\ln x}$, for $x > \mathrm{e}^{-2}$.

(b) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(c) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.